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Ideal Bose Gas in Higher Dimensions

غاز بوز مثالي في ابعاد عالية

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Abstract

Some physical properties of the behavior of an ideal non- relativistic Bose gas in N – dimensional space are theoretically investigated. The general analytic expressions of the critical temperature T_c of Bose – Einstein condensation, and the high- temperature behavior of the gas have been derived. The dependence of these physical quantities on space dimension is discussed and some numerical values are calculated. Moreover, the limit of these quantities in the infinite dimensional space ($N \rightarrow \infty$) is also examined.

Key words: Bose gas, Bose- Einstein condensation, higher dimensions.

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ملخص

تم بحث الخواص الفيزيائية النظرية لغاز بوز غير النسبي في فراغات متعددة الابعاد. فقد تم اشتقاق تعابير تحليلية لدرجة الحرارة الحرجة لحدوث تكثف بوز – آينشتاين ولتصرف الغاز عند درجات حرارة عالية.

تمت مناقشة اعتماد هذه الكميات الفيزيائية على البعد الفراغي وتم حساب بعض القيم العددية. إضافة إلى ذلك فقد تم التمعن في هذه الكميات عندما يؤول البعد الفراغي الى كمية غير نهائية.

1. Introduction

Recently, the study of the ideal Bose gas has been extensively carried out by many researchers (Chen and Lin, 2003; Leggett, 2001; Edery, 2006; Tomas, 2006; Ovchinnikov, 1993). Bose gases in confined geometries has been investigated (Blakie, 2005; Hao et. al ., 2006; Salasnich , 2004). The study of Bose – Einstein condensation has long been of wide interest by many workers (Galli, et al, 2005; Mayers, 2006; Kim et al., 1999; Algin and Deviren, 2005).

Physical problems in higher dimensions have been widely investigated. Besides its mathematical interest (Belhaj and Rasmussen, 2005; Golovnev, 2006), the N- dimensional space has been used in the study of Schr dinger equation with different kinds of potentials (Dong and Sun, 2004; Quan et al., 2005; Znojil, 2000; Romera et al., 2006).

Some workers investigated higher dimensional gravity (Troncoso and Zanelli, 2000; Tanabe et al., 2006). Others examined rotating black holes in multidimensional space (Gibbons et al., 2004; Rogatko, 2000; Berti et al., 2006; Kunduti et al., 2006). Furthermore, gravitational collapse in higher dimensions has been of much interest to investigators (Ilha et al., 1999; Patil, 2003; Goswami et al., 2004). Some researchers discussed casimir energy (Huang, 2000; Fabi et al., 2006) and others explored unified theories (Gogoladze et al., 2003; Gogoladze et al., 2005) in N- dimensional space. The development of techniques to trap and cool atoms and the experimental achievements of Bose- Einstein condensation have stimulated great interest in the theoretical study of Bose gas in higher dimensions. For example (Yan, 2000) derived the equation of state of an ideal Bose gas trapped in n- dimensional generic power- law potential. (Standen and Toms, 1998) discussed Bose – Einstein condensation of the magnetized Bose gas and showed that for large values of the magnetic field the gas undergoes a dimensional reduction. Others (Kolomeisky et al., 1992; Crisano et al., 2002) examined renormalization – group analysis of the dilute Bose gas in *N* dimensions.

In addition, Lozovik et al. (1985) showed that the threshold interaction constant for the existence of a condensed phase increases with the spatial dimension of a Bose system.

Therefore, we believe that it is imperative to study the extent of the dimensionality contribution to the physical properties of a Bose gas.

In this context, some properties of the ideal Bose gas in higher dimensions are investigated. In section 2, the particle number and the energy densities are derived. In section 3, an expression relating the constant $\alpha\ (= -\mu / KT)$ with temperature and space dimension will be derived. In section 4, the researcher is going to consider Bose- Einstein condensation and show the dependence of the critical temperature on the dimension *N*. Section 5, is devoted to the discussion and results.

2. The Ideal Bose Gas in N Dimensions

We consider a system of non- interacting Bose gas confined within a cube with sides of length L and impenetrable walls at a finite temperature. The number of particles n (E) dE within the energy interval E to E+ dE is

$$\text{n(E) dE} = \text{D(E)}\ \left[e^{\beta E + a} - 1\right]^{-1} dE \qquad (1)$$

where D(E) is the number of single – particle states per unit energy interval. The constant $\alpha(= -\beta\mu)$ is related to the fugacity Z as Z $= e^{-\alpha}$, with $\beta = 1/kT$, k is the Boltzmann constant , and μ is the chemical potential. The function D (E) in N dimensions (Al- Jaber , 1999) is given by

$$D(E) = (N-1)\left(\frac{L}{2\pi}\right)^{N} \frac{\pi^{N/2}}{\Gamma\left(N/2\right)} \left(\frac{2m}{\hbar^{2}}\right)^{N/2} E^{(N-2)/2} \qquad (2)$$

Therefore, the number of particles per unit volume within the range $(E, E + dE)$ is

$$\frac{n(E)\,dE}{V}=\frac{n(E)\,dE}{L^N}=$$

$$\frac{(N-1)\left(2m/\hbar^2\right)^{N/2}}{2^N\ \pi^{N/2}\Gamma\left(N/2\right)}\ E^{(N-2)/2}\ \left[e^{\beta E+\alpha}-1\right]^{-1}dE, \tag{3}$$

and thus the number density ρ of the particles is

$$\rho=\frac{1}{V}\int_0^\infty n(E)dE=\frac{(N-1)\left(2m/\hbar^2\right)^{N/2}}{2^N\ \pi^{N/2}\Gamma\left(N/2\right)}\quad\int_0^\infty\frac{E^{(N-2)/2}\ dE}{\left(e^{\beta E+\alpha}-1\right)} \tag{4}$$

The total energy per unit volume, U, of the gas is:

$$U=\frac{1}{V}\int_0^\infty E\ n(E)\,dE$$

$$=\frac{(N-1)\left(2m/\hbar^2\right)^{N/2}}{\pi^{N/2}\,\Gamma\left(N/2\right)}\int_0^\infty\frac{E^{N/2}dE}{e^{\beta E+\alpha}-1} \tag{5}$$

The High – Temperature Limit

When e^{α} >>1, we can neglect the 1 in (4) and we get

$$\rho=\frac{(N-1)\left(2m/\hbar^2\right)^{N/2}}{2^N\,\pi^{N/2}\,\Gamma\left(N/2\right)}\int_0^\infty E^{(N-2)/2}e^{-(\beta E+\alpha)}dE \tag{6}$$

The integral in eq.(6) is

$$\int_0^\infty E^{(N-2)/2}e^{-\beta E}dE=\frac{\Gamma(N/2)}{\beta^{N/2}}, \tag{7}$$

and thus

$$\rho=\frac{(N-1)}{2^N\ \pi^{N/2}\ \beta^{N/2}}\left(\frac{2m}{\hbar^2}\right)^{N/2}e^{-\alpha}$$

which can be written as

$$\rho = (N-1)\left(\frac{mkT}{2\pi\hbar^2}\right)^{N/2} e^{-\alpha}, \qquad (8)$$

and it yields the well- known result in the three dimensional space

(N =3), (Bransden and Jochain, 2000). Equation (8) can be solved for α as:

$$e^{\alpha} = \frac{N-1}{\rho}\left(\frac{m\mathrm{k}}{2\pi\hbar^2}\right)^{N/2} T^{N/2} \qquad (9)$$

It is clear that the constant α depends on the particle density ρ, the temperature T , and the dimension *N*. Introducing the mean thermal wavelength

$\lambda = h/(2\pi m kT)^{1/2}$, (10)

we can rewrite equation (9) as

$e^{\alpha} = (N-1)/(\rho\lambda^N)$ (11)

For numerical values, we consider ^4He whose mass density is 0.15 x 10^3kg/m^3, so that the particle density ρ is

$$\rho = \frac{mass/Vol.}{mass/atom} = \frac{0.15\times10^3}{4(1.67\times10^{-27})} = 2.2\times10^{28}\ m^{-3}$$

and the quantity $\frac{mk}{2\pi\hbar^2} = 1.33\times10^{18}\ m^{-2}k^{-1}$

Table I shows numerical values of α at a given temperature for different values of the dimension *N*.

It is noticed that for a given temperature, the constant α increases as the dimension *N* increases, which means that the Bose gas becomes closer and closer to Maxwell – Boltzmann gas as the dimension increases. This is equivalent to the condition ($\rho\lambda$N) 1 that expresses the low-particle density and the high temperature limit of the gas. In this case, α increases and hence the chemical potential μ becomes negative with large magnitude and thus the fugacity, z(=exp(μ/kT)) of the system must be much smaller than unity. This shows that quantum effects due to identity of particles become less important in higher dimensions. It is interesting to investigate the limit as $N \to \infty$; we can rewrite (9) as:

$$\alpha = \frac{N}{2} \ln \left[\frac{mkT}{2\pi \hbar^2} \left(\frac{N-1}{\rho} \right)^{\frac{2}{N}} \right]. \tag{12}$$

We note $\lim\limits_{N \to \infty} \left(\frac{N-1}{\rho} \right)^{\frac{2}{N}} = 1$, and hence

$$\lim_{N\, l\arg e} \alpha = \frac{N}{2} \ln \left[\frac{mkT}{2\pi \hbar^2} \right], \tag{13}$$

For example, for T = 10^{-18} K, and N = 1000 the value of α for ^{4}He would be 142.6.

Table (1): Variation of α with N at a given Temperature.

T= 300 K		T =100 K	
N	α	**N**	α
3	6.58	3	4.93
4	30.7	4	28.5
5	54.7	5	51.9
6	78.6	6	75.3
7	102.5	7	98.7
8	126.4	8	122.0
9	150.3	9	145.3
10	174.1	10	168.6
11	197.9	11	191.9
12	221.7	12	215.1

One may further examine the variation of temperature with the dimension N for a given value of α . To that end, we solve (12) for T;

$$T = \frac{2\pi\hbar^2}{mk}\left[\frac{\rho e^{\alpha}}{N-1}\right]^{\frac{2}{N}}, \qquad (14)$$

and take α = 6.58 (which is its value for ^{4}He in the three dimensional space) . This is shown in Table (II), which clearly indicates that the temperature decreases dramatically as N increases. Furthermore, the last few entries in the table show that the temperature reaches a minimum value for a very large N. This is obvious from (14), since $\left(\frac{\rho e^{\alpha}}{N-1}\right)^{\frac{2}{N}} \xrightarrow[N\,large]{} 1$, and hence $T = \xrightarrow[N\,large]{} \frac{2\pi\hbar^2}{mk}$ whose value, for ^{4}He at α = 6.58, is 7.518 X 10^{-19}K.

This demonstrates that the absolute zero is really a hypothetical theoretical value, at least in this context.

Table (2): Variation of T with N at a given α .

N	T	N	T
3	299	20	$7.3X10^{-16}$
4	$1.7X10^{-3}$	30	$7.2X10^{-17}$
5	$1.3X10^{-6}$	50	$1.13X10^{-18}$
6	$1.1X10^{-8}$	60	$8.8X\ 10^{-18}$
7	$3.7X10^{-10}$	70	$5.1X10^{-18}$
8	$2.9X10^{-11}$	100	$2.8X10^{-18}$
9	$4.1X10^{-12}$	200	$1.8X10^{-18}$
10	$8.4X10^{-13}$	400	$1.0X10^{-18}$
11	$2.3X10^{-13}$	500	$9.7X10^{-19}$
12	$8X10^{-14}$	10^4	$7.6X10^{-19}$
13	$3.2X10^{-14}$	10^5	$7.5X10^{-19}$
16	$4.2X10^{-15}$	10^{10}	$7.518X10^{-19}$

4. Bose – Einstein Condensation

In the low-temperature, high density limit D (0) = 0, but because there is one state at E=0, D (E) must be modified

D (E) = D (E) + δ (E), and thus (4) becomes

$$\rho = \rho_\circ + \frac{(N-1)\left(2m/\hbar^2\right)^{N/2}}{2^N \pi^{N/2}\Gamma(N/2)} \int_0^\infty \frac{E^{(N-2)/2}}{e^{\beta E+\alpha}-1}\, dE \tag{15}$$

where $\rho_\circ$ is the particle density in the ground state with E= 0. For bosons, any number of particles can occupy the single – particle level of lowest energy; therefore, there is no limitation on the size of the particle densities $\rho_\circ$ or ρ. It is interesting to find the temperature below which the particles are forced to condense into the ground state.

Let x= $\beta E = E/kT$, so that (15) becomes

$$\rho - \rho_\circ = \frac{(N-1)\left(2mkT/\hbar^2\right)^{N/2}}{2^N \pi^{N/2}\Gamma(N/2)} \int_0^\infty \frac{x^{(N-2)/2}dx}{e^{x+\alpha}-1} \tag{16}$$

The fraction of particles in states other than the ground state, $(\rho-\rho_0)/\rho$ is thus

$$\frac{\rho-\rho_\circ}{\rho} = \frac{(N-1)}{2^N \pi^{N/2}\Gamma(N/2)} \frac{1}{\rho}\left(\frac{2mkT}{\hbar^2}\right)^{N/2} f_{(N-2)/2}(\alpha) \tag{17}$$

Where

$$f_n(\alpha) = \int_0^\infty \frac{x^n\, dx}{e^{x+\alpha}-1} \tag{18}$$

The largest possible value of f (α) occurs when α=0 (since, $\alpha \geq 0$).

$$f_{(N-2)/2}(0) = \int_0^\infty \frac{x^{(N-1)/2}dx}{e^x-1} = \Gamma\left(\frac{N}{2}\right)\xi\left(\frac{N}{2}\right)$$

Where $\xi(n)$ is the Reimann zeta function. When the function $(\rho-\rho_0)/\rho$ becomes less than unity, the system condenses into the ground state. This occurs at temperatures T<T_c (called critical temperature), where

$$T_c = \frac{\hbar^2}{2mk}\left[\frac{2^N \pi^{N/2}\rho}{(N-1)\xi\left(\frac{N}{2}\right)}\right]^{2/N} = \frac{2\pi\hbar^2}{mk}\left[\frac{\rho}{(N-1)\xi\left(\frac{N}{2}\right)}\right]^{2/N} \tag{19}$$

It is interesting to note that as the dimension N increases, T_c decreases. This means that the critical temperature at which the system starts to condensate shifts towards lower values as the dimension *N* increases. It is illustrative to use the relevant data for ^4He to calculate the critical temperature for different values of the space dimension. This is shown in table (III) below.

Table (3): Variation of critical temperature with *N*.

N	T_c	N	T_c
3	1.96	20	$3.8X10^{-16}$
4	$5.0X10^{-5}$	30	$4.6\ X10^{-17}$
5	$8.2X10^{-8}$	40	$1.6X10^{-17}$
6	$1.2X10^{-9}$	50	$8.7X10^{-18}$
8	$5.5\ X10^{-12}$	100	$2.5\ X\ 10^{-18}$
10	$2.2\ X\ 10^{-13}$	200	$1.3X10^{-18}$
12	$2.6\ X\ 10^{-14}$	1000	$8.4\ X10^{-19}$
14	$5.8\ X\ 10^{-15}$	10^4	$7.6\ X\ 10^{-19}$
16	$1.8\ X\ 10^{-15}$	10^{10}	$7.519X10^{-19}$

It is tempting to consider the limit as $N \to \infty$. In this case $\xi\left(\frac{N}{2}\right) \xrightarrow[N\to\infty]{} 1$, and thus $\lim_{N\to\infty} T_c = \frac{10^{-18}}{1.33} = 7.5188 \times 10^{-19}\ K$.

5. Discussion and Results

In this paper, we considered some physical properties of the ideal Bose gas in N- dimensional space. In particular, the particle number and energy densities were derived. In the high temperature limit, the constant $\alpha = (= -\mu / kT)$ with μ being the chemical potential was found to depend on the particle number, temperature, and the dimension N . As an illustration, we calculated numerical values for the constant α, in the classical limit, at a given temperature, at different values of the dimension N , and it is found that α increases as N increases. In addition, for a given α we presented numerical results for the temperature of the gas in different dimensions and showed that it decreases as N increases and reaches a minimum value in the infinite dimensional space.

Furthermore, we analyzed the Bose- Einstein condensation in higher dimensions and derived the critical temperature as function of the particle number density and the dimension *N*. It was concluded that the critical temperature decreases as the dimension increases and it reaches a minimum value of about 7.52 X 10^{-19} k for ^{4}He in the infinite dimensional space, $(N \to \infty)$.

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